Distribution of current in nonequilibrium diffusive systems and phase transitions

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We consider diffusive lattice gases on a ring and analyze the stability of their density profiles conditionally to a current deviation. Depending on the current, one observes a phase transition between a regime where the density remains constant and another regime where the density becomes time dependent. Numerical data confirm this phase transition. This time dependent profile persists in the large drift limit and allows one to understand on physical grounds the results obtained earlier for the totally asymmetric exclusion process on a ring.

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I. INTRODUCTION

Stochastic lattice gas models have been extensively studied recently as they are among the simplest examples of nonequilibrium systems. They describe particles evolving on a lattice according to some stochastic dynamics which do not satisfy detailed balance [1,2]. For example, the lattice gas dynamics might satisfy detailed balance in the bulk dynamics but not at the boundaries, in such a way that it mimics a diffusive system in contact with reservoirs at unequal densities or temperatures [3–6]. The detailed balance might also be broken in the bulk, for example to represent the effect of a driving field [7,8].

A powerful approach to understand the steady state of nonequilibrium systems was developed by Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim, as a macroscopic fluctuation theory (MFT) which gives, for large diffusive systems, the probability distribution of trajectories in the space of density profiles [3,4]. The MFT relies on the hydrodynamic large deviation theory [2,8,9] which provides estimates for the probability of observing atypical space and time dependent density profiles. It gives a framework to calculate a large number of properties of stochastic lattice gas, such as the large deviation functional of the density profiles. Recent developments of the hydrodynamic large deviation theory [10,11] allowed one to estimate also the large deviations of the current through the system. What the MFT provides is a variational principle from which one can write the equations of the time evolution of the most likely density profile responsible of a given fluctuation. What it does not provide, in general, is the solution of these equations which would give quantitative predictions for the distribution of the fluctuations. So far, for the large deviation functional of the density profiles in the steady state, the equations could only be solved in a few cases of nonequilibrium systems with open boundaries (the symmetric simple exclusion process [4], the Kipnis, Marchioro, Presutti model [12,13]). For the symmetric simple exclusion process (SSEP) the results of the MFT were in full agreement with the results obtained [5,6,14] from the exact knowledge of the weights of the microscopic configurations in the steady state.

In our previous work [11], we developed a theory to calculate the large deviation function of the current through a long one dimensional diffusive lattice gas in contact at its two ends with two reservoirs at unequal densities. Our approach was based on an assumption, the *additivity principle*, which relates the large deviation function of the current of a system to the large deviation functions of subsystems, when one breaks a large system into large subsystems. This assumption is in fact equivalent to the hypothesis, within the hydrodynamic large deviation framework, that to observe, for a very long time period T, an average current $q=Q_T/T$, the system adopts a profile with a shape, fixed in time, but of course depending on q (here Q_T is the total number of particles transferred, say from the left reservoir to the system during time T). The additivity principle allows one to obtain explicit expressions [11] for all the cumulants of the integrated current Q_T. The predictions of our theory were tested in a few cases [11] and the results were found in complete agreement with what was already known or what could be derived by alternative approaches [15–17].

Recently, it was pointed out [10] that even if our predictions [11] are valid for some diffusive lattice gas, it might happen that, to produce an average current q over a long period of time, the best profile is time dependent. The optimal profile is a solution of a variational principle [10] and one needs to optimize over the space and time dependent profiles to determine whether or not the optimal profile is time dependent. One of the goals of this paper is to analyze this variational principle for lattice gas on a ring and to show that depending on the parameters, the optimal profile may or not be time dependent. This can be interpreted as a dynamical phase transition.

Our paper is organized as follows. In Sec. II, we recall the variational principle which allows one to calculate the large deviations of the current in diffusive lattice gas. In Sec. III, we consider a general lattice gas on a ring and show under what conditions the flat profile becomes unstable. In Sec. IV, we write the large deviation function of the current, when the optimal profile has a fixed shape moving at a constant velocity. In Sec. V, we present exact numerical results on the weakly asymmetric exclusion process for small system sizes which give evidence that for some range of parameters, consistent with the results of Sec. IV, the optimal profile is no longer flat but becomes space-time dependent (for $\rho_0=1/2$ it is only space dependent). In Sec. VI, we analyze the limit of a strong asymmetry and obtain a simple expression for the large deviation function of the current, under the assumption that the optimal profile becomes in this limit a step function. In Sec. VII, we show that, even for a strongly asymmetric case such as the totally asymmetric exclusion process, one can exhibit time-dependent profiles determined by the Jensen Varadhan functional [18] which give, for large system size, the exact large deviation function of the current previously calculated by the Bethe ansatz.

II. MACROSCOPIC FLUCTUATION THEORY

Let us consider, as we shall do it in the rest of this paper, the time evolution of a one dimensional stochastic lattice gas on a lattice of N sites. According to the hydrodynamic formalism [2], a given diffusive lattice gas can be characterized by two functions $D(\rho)$ and $\sigma(\rho)$ of its density ρ. One way to define them [11] is to consider a one dimensional system of length N connected to reservoirs at its two ends. For such a lattice gas, the variance of the total charge Q_T transferred during a long time T from one reservoir to the other is given, for large N, by definition of $\sigma(\rho)$ by

$$\frac{\langle Q_T^2\rangle}{T} = \frac{\sigma(\rho)}{N} \tag{1}$$

when both reservoirs are at the same density ρ. On the other hand, if the left reservoir is at density $\rho+\Delta\rho$ and the right reservoir is at density ρ, the average current is given, for small $\Delta\rho$, by

$$\frac{\langle Q_T\rangle}{T} = \frac{D(\rho)\Delta\rho}{N} \tag{2}$$

which is simply Fick's law and defines the function $D(\rho)$. In the symmetric simple exclusion process $\sigma(\rho)=\rho(1-\rho)$ and $D(\rho)=1/2$ [2] whereas in the Kipnis, Marchioro, and Presutti model [12,13] $\sigma(\rho)=\rho^2$ and $D(\rho)=1/2$. The effect of a uniform weak electric field of strength $\nu/(2N)$ acting from left to right on the particles is to modify Eq. (2) into

$$\frac{\langle Q_T\rangle}{T} = \frac{D(\rho)\Delta\rho}{N} + \frac{\nu\sigma(\rho)}{N}. \tag{3}$$

This equation follows from the linear response theory [2].

The coefficients $D(\rho)$ and $\sigma(\rho)$ are sufficient to characterize a diffusive system at a macroscopic scale and on macroscopic times (see the Appendix). In particular, the probability of observing the evolution of a density profile $\rho(x,s)$ and a rescaled current $j(x,s)$ for $0<s<T/N^2$ during a microscopic time $T\sim N^2$ is given, according to the hydrodynamic large deviation theory [10], by

$$\text{Pro}(\{j(x,s),\rho(x,s)\}) \sim \exp[-N\,\mathcal{I}^{\nu}_{[0,T/N^2]}(j,\rho)], \tag{4}$$

where $\mathcal{I}^{\nu}_{[0,t]}$ is defined by

$$\mathcal{I}^{\nu}_{[0,t]}(j,\rho) = \int_0^t ds \int_0^1 dx \frac{[j(x,s) + D(\rho(x,s))\rho'(x,s) - \nu\sigma(\rho(x,s))]^2}{2\sigma(\rho(x,s))} \tag{5}$$

with $\rho'=\partial\rho/\partial x$ and where the rescaled current $j(x,s)$ is related to the density profile $\rho(x,s)$ by the conservation law

$$\frac{d\rho(x,s)}{ds} = -\frac{dj(x,s)}{dx}. \tag{6}$$

A formalism equivalent to this hydrodynamic large deviation theory was developed [19,20] in the context of the full counting statistics of the transport of free fermions through disordered wires. A simple derivation of Eqs. (5) and (6) is given in the Appendix.

The large deviation function $G(j_0)$ of the current is then defined as

$$\text{Pro}\left(\frac{Q_T}{T} = \frac{j_0}{N}\right) \sim \exp\left(\frac{T}{N}G(j_0)\right) \text{ for large } T \text{ and } N \tag{7}$$

(in Eq. (7), one has first to take the limit $T\to\infty$ and then make N large; in practice Eq. (7) should hold when $T \gg N^2$ as N^2 is the characteristic time of a diffusive system of size N).

Now according to Eq. (4), the large deviation function $G(j_0)$ is given by

$$G(j_0) = \lim_{t\to\infty}\left[-\frac{1}{t}\min_{\rho(x,s)}\mathcal{I}^{\nu}_{[0,t]}(j,\rho)\right], \tag{8}$$

where the current $j(x,s)$ satisfies for large t and all x the constraint

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t j(x,s)ds = j_0 \tag{9}$$

with the profile $\rho(x,s)$ and the current $j(x,s)$ connected by Eq. (6).

In the following, we will often consider, instead of Eq. (7), the generating function of the current:

$$\langle e^{\lambda Q_T}\rangle \sim e^{T\mu(\lambda)} \text{ for large } T \tag{10}$$

and then, according to Eqs. (7) and (8), $\mu(\lambda)$ is given by

$$\begin{aligned}\mu(\lambda) &= \frac{1}{N}\max_{j_0}[\lambda j_0 + G(j_0)] \\ &= \frac{1}{N}\lim_{t\to\infty}\frac{1}{t}\max_{\rho(x,s)}\left[\lambda\int_0^t j(x,s)ds - \mathcal{I}^{\nu}_{[0,t]}(j,\rho)\right],\end{aligned} \tag{11}$$

where the maximum is taken over all the density and current evolutions such that the conservation law (6) is satisfied.

For a system of length N connected to two reservoirs at densities ρ_a and ρ_b at its two ends, the calculation of the large deviation function $G(j_0)$ of the current is therefore reduced to finding the time-dependent profile $\rho(x,s)$ which optimizes Eq. (8) under the constraints (6) and (9) and with the additional boundary conditions $\rho(0,s)=\rho_a$ and $\rho(1,s)=\rho_b$. All the results of our previous work [11] follow then from the assumption that this optimal profile does not vary with time (except from boundary effects near time 0 and time t which do not contribute in the large t limit).

For a system on a ring of N sites, as we shall consider in the present work, the optimization problem is the same except for the boundary condition which becomes $\rho(0,s)=\rho(1,s)$ and the fact that the total density ρ_0 on the ring becomes an additional conserved quantity

$$\int_0^1 \rho(x,s)dx = \rho_0.$$

III. CONDITIONS FOR THE STABILITY OF A FLAT PROFILE

In this section, we consider a lattice gas on a ring of N sites with total density ρ_0 and we determine the condition for the flat profile (uniformly equal to ρ_0) to be the optimal profile in Eqs. (8) or (11).

Under the assumption that the optimal profile in Eq. (8) is

$$\rho(x,s) = \rho_0$$

the large deviation function $G(j_0)$ is given [Eqs. (5) and (8)] by

$$G_{\text{flat}}(j_0) = -\frac{[j_0 - \nu\sigma(\rho_0)]^2}{2\sigma(\rho_0)} \tag{12}$$

which, by Eq. (11), gives for $\mu(\lambda)$

$$\mu_{\text{flat}}(\lambda) \simeq \frac{\lambda(\lambda + 2\nu)\sigma(\rho_0)}{2N}. \tag{13}$$

A natural question is whether one could increase $G(j_0)$ [and $\mu(\lambda)$] by adding to this flat profile some small ($\epsilon \ll 1$) space and time dependent perturbation of the form

$$j(x,s) = j_0 + \epsilon[j_1(x)\cos(\omega s) + j_2(x)\sin(\omega s)]$$

in which case due to Eq. (6)

$$\rho(x,s) = \rho_0 + \frac{\epsilon}{\omega}[-j_1'(x)\sin(\omega s) + j_2'(x)\cos(\omega s)],$$

where $j_1(x)$ and $j_2(x)$ are periodic functions of period 1. The resulting expression (8) for $G(j_0)$ to second order in ϵ is

$$G(j_0) = -\frac{[j_0 - \nu\sigma(\rho_0)]^2}{2\sigma(\rho_0)} + \epsilon^2\int_0^1 dx\Bigg[-\frac{j_1^2 + j_2^2}{4\sigma(\rho_0)} - \frac{(j_1''^2 + j_2''^2)D(\rho_0)^2}{4\omega^2\sigma(\rho_0)} + \frac{j_0(j_1j_2' - j_2j_1')\sigma'(\rho_0)}{2\omega\sigma^2(\rho_0)} + (j_1'^2 + j_2'^2)\left(\frac{j_0^2\sigma''(\rho_0)}{8\omega^2\sigma(\rho_0)^2} - \frac{\nu^2\sigma''(\rho_0)}{8\omega^2} - \frac{j_0^2\sigma'_2(\rho_0)}{4\omega^2\sigma^3(\rho_0)}\right)\Bigg].$$

As this expression is quadratic in the currents j_1 and j_2, the various Fourier modes are not coupled. Choosing for $j_1(x)$ and $j_2(x)$

$$j_1(x) = a\cos(2\pi x) + b\sin(2\pi x),$$

$$j_2(x) = c\cos(2\pi x) + d\sin(2\pi x), \tag{14}$$

one gets

$$G(j_0) = -\frac{[j_0 - \nu\sigma(\rho_0)]^2}{2\sigma(\rho_0)} + \epsilon^2\Bigg[(ad - bc)\frac{j_0\sigma'(\rho_0)\pi}{\omega\sigma^2(\rho_0)} - (a^2 + b^2 + c^2 + d^2)\left(\frac{1}{8\sigma(\rho_0)} + \frac{2\pi^4D^2(\rho_0)}{\omega^2\sigma(\rho_0)} + \frac{\pi^2\sigma'^2(\rho_0)j_0^2}{2\omega^2\sigma^3(\rho_0)} + \frac{\nu^2\pi^2\sigma''(\rho_0)}{4\omega^2} - \frac{\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2\sigma^{2\prime}(\rho_0)}\right)\Bigg].$$

The flat profile is stable against the perturbation (14) if $-G(j_0)$ is a positive definite quadratic form in a,b,c,d. This is achieved when for all ω

$$\frac{1}{8\sigma(\rho_0)} + \frac{2\pi^4D^2(\rho_0)}{\omega^2\sigma(\rho_0)} + \frac{\pi^2\sigma'^2(\rho_0)j_0^2}{2\omega^2\sigma^3(\rho_0)} + \frac{\nu^2\pi^2\sigma''(\rho_0)}{4\omega^2} - \frac{\pi^2\sigma''(\rho_0)j_0^2}{4\omega^2\sigma^2(\rho_0)} > \left|\frac{j_0\sigma'(\rho_0)\pi}{2\omega\sigma^2(\rho_0)}\right|. \tag{15}$$

The flat profile becomes therefore unstable if for at least one ω the inequality (15) is not satisfied, i.e.,

$$8\pi^2D^2(\rho_0)\sigma(\rho_0) + [\nu^2\sigma^2(\rho_0) - j_0^2]\sigma''(\rho_0) < 0 \tag{16}$$

which, given Eqs. (11)–(13), can be rewritten as

$$4\pi^2D^2(\rho_0) < N\mu_{\text{flat}}(\lambda)\sigma''(\rho_0). \tag{17}$$

The first two frequencies to become unstable are $\omega=2\pi j_0\sigma'(\rho_0)/\sigma(\rho_0)$ [with the amplitudes in Eq. (14) such that $a=d$ and $b=-c$] and $\omega=-2\pi j_0\sigma'(\rho_0)/\sigma(\rho_0)$ (with $a=-d$ and $b=c$). As we are left with two arbitrary amplitudes (a and b), the corresponding current is determined up to two constants A and x_0

$$j(x,t) = j_0 + A\cos\left[2\pi\left(x - x_0 - \frac{j_0\sigma'(\rho_0)}{\sigma(\rho_0)}t\right)\right]. \tag{18}$$

One should be able to determine the amplitude A by expanding $G(j_0)$ to higher order in ϵ whereas the parameter x_0 should remain undetermined due to the translation invariance of the problem.

One could analyze in a similar way the stability of the flat profile against other modes by choosing $j_1(x)=a\cos(2\pi nx)+b\sin(2\pi nx)$ and $j_2(x)=c\cos(2\pi nx)+d\sin(2\pi nx)$ and the threshold (16) would become

$$8\pi^2D^2(\rho_0)n^2\sigma(\rho_0) + [\nu^2\sigma^2(\rho_0) - j_0^2]\sigma''(\rho_0) < 0.$$

This shows that the fundamental ($n=1$) is the first mode to become unstable.

IV. A SIMPLE TIME DEPENDENT PROFILE

The form (18) suggests that beyond the instability the optimal profile is a fixed shape moving at a constant velocity v,

$$\rho(x,t) = g(x - vt). \tag{19}$$

In this section we are going to provide an expression for the optimal g.

Due to conservation law (6) the current is then

$$j(x,t) = j_0 - v\rho_0 + vg(x - vt).$$

If such a profile is the optimal profile, then the variational principle (8) reduces to

$$G(j_0) = -\min_{g(x),v} \int_0^1 \frac{dx}{2\sigma(g(x))}[j_0 - v\rho_0 + vg(x) + D(g(x))g'(x) - \nu\sigma(g(x))]^2. \tag{20}$$

If one expands the square (the terms linear in g' give a null contribution due to the periodic boundary conditions), one gets

$$G(j_0) = -\inf_{g(x),v} \int_0^1 dx[X(g) + g'^2 Y(g)], \tag{21}$$

where

$$X(g) = \frac{[j_0 - v\rho_0 + vg - \nu\sigma(g)]^2}{2\sigma(g)}$$

and

$$Y(g) = \frac{D^2(g)}{2\sigma(g)}.$$

The optimal v in Eq. (20) is then given by

$$v = -\frac{\int dx \frac{(g-\rho_0)(j_0 - \nu\sigma(g))}{\sigma(g)}}{\int dx \frac{(g-\rho_0)^2}{\sigma(g)}} = -j_0 \frac{\int dx \frac{(g-\rho_0)}{\sigma(g)}}{\int dx \frac{(g-\rho_0)^2}{\sigma(g)}}, \tag{22}$$

this last simplification being due to the constraint $\int g(x)dx = \rho_0$. With this constraint and for a fixed v, a variational calculation of the optimal g in Eq. (21) shows that g should satisfy

$$X'(g) - 2Y(g)g'' - g'^2 Y'(g) = C_2.$$

Multiplying both sides by g' allows one to integrate once so that g satisfies

$$X(g) - g'^2 Y(g) = C_1 + C_2 g, \tag{23}$$

where C_1 and C_2 are constants [which is an extension of Eq. (15) of Ref. [11] to the case of the ring].

For fixed j_0, ρ_0, and v, if one denotes by g_1 and g_2 the two extrema of the profile g [generically, the profile $g(x)$ is a periodic function of period 1 with a single minimum g_1 and a single maximum g_2], one can determine the constants C_1 and C_2 by Eq. (23) in terms of g_1 and g_2 [as $X(g_1) = C_1 + C_2 g_1$ and $X(g_2) = C_1 + C_2 g_2$]. The differential equation (23) determines the whole profile (up to a translation on the ring) and the constants g_1 and g_2 are then fixed by the fact that

$$\frac{1}{2} = \int_{x(g_1)}^{x(g_2)} dx = \int_{g_1}^{g_2} \frac{dg}{g'} = \int_{g_1}^{g_2} \sqrt{\frac{Y(g)}{X(g) - C_1 - C_2 g}} dg$$

and

$$\frac{\rho_0}{2} = \int_{g_1}^{g_2} g \sqrt{\frac{Y(g)}{X(g) - C_1 - C_2 g}} dg.$$

V. EXACT NUMERICS FOR THE WEAKLY ASYMMETRIC EXCLUSION PROCESS ON A RING

In order to support the analytical calculations of Sec. IV, we wrote a program to calculate exactly $\mu(\lambda)$ for the weakly asymmetric simple exclusion process (WASEP) on a ring of N sites with $P = N\rho$ particles. In the simple symmetric exclusion process (SSEP), each particle jumps to its right at rate $\frac{1}{2}$ and to its left at rate $\frac{1}{2}$ and the functions $D(\rho)$ and $\sigma(\rho)$ are given [2] by

$$D_{\mathrm{SSEP}} = \frac{1}{2}, \tag{24}$$

$$\sigma_{\mathrm{SSEP}} = \rho(1 - \rho). \tag{25}$$

If one introduces a weak electric field to the right, the model becomes the WASEP and the rates become $\frac{1}{2} + \nu/2N$ to the right and $1/2 - \nu/2N$ to the left.

As the evolution is a Markov process, one can build, as explained in Refs. [21,22] from the Markov matrix, a λ-dependent matrix, the largest eigenvalue of which is $\mu(\lambda)$ defined by Eq. (10). According to the linear stability analysis, the flat profile becomes unstable [Eq. (16)] for

$$j_0^2 < \rho(1 - \rho)[\nu^2 \rho(1 - \rho) - \pi^2]$$

or by Eq. (17) for

$$N\mu_{\mathrm{flat}}(\lambda) < -\frac{\pi^2}{2}. \tag{26}$$

We have calculated the exact eigenvalue $\mu(\lambda)$ for lattice sizes from $N = 6$ to 22, at density 1/2 for an asymmetry $\nu = 10$ (in order to avoid negative rates for small system sizes, we have replaced in our programs the rates $\frac{1}{2} \pm \nu/2N$ by $\exp[\pm\nu/N]/2$). The results in Fig. 1 show a rather quick convergence with increasing N towards the value corresponding to a shape determined by Eq. (20). Clearly the flat profile gives a value too low, incompatible with the numerical data.

VI. A BRIDGE BETWEEN A WEAK AND A STRONG ASYMMETRY

In this section, we consider the large ν limit and derive the asymptotic large deviation functional as well as the limit of the optimal profiles discussed in Sec. IV.

We assume that the optimal profile is the traveling wave (19) and we consider current deviations of the form

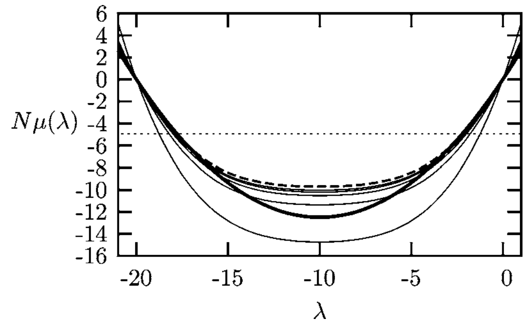

FIG. 1. $\mu(\lambda)$ as defined by Eq. (10) for the weakly asymmetric exclusion process on a ring of N=6, 10, 14, 18, 22 sites when the asymmetry parameter $\nu=10$ and the density 1/2 (thin lines). As N increases, the data increase and seem to accumulate to the value predicted (dashed line) by assuming that the optimal profile is the one discussed in Sec. IV. If the optimal profile had been flat, the curve would have been predicted by Eq. (13) (thick line). The horizontal dotted line gives the value of μ below which the flat profile becomes unstable [Eq. (26)].

$$j_0 = \nu i_0. \tag{27}$$

Then for large ν, the moving profile which satisfies (23) becomes very steep in the regions where it varies. This is illustrated for the WASEP in Fig. 2 where several traveling waves are represented for increasing values of ν (with i_0=0). More precisely, the optimal profile takes the form of a step function with two constant values g_1 and g_2 separated by two discontinuities,

$$g(x) = g_1 \text{ for } 0 < x < y,$$

$$g(x) = g_2 \text{ for } y < x < 1, \tag{28}$$

so that the parameters g_1, g_2, and y are related to ρ_0 and i_0 by

$$\rho_0 = y g_1 + (1-y) g_2, \tag{29}$$

$$i_0 = y\sigma(g_1) + (1-y)\sigma(g_2). \tag{30}$$

The expression of the velocity (22) then becomes

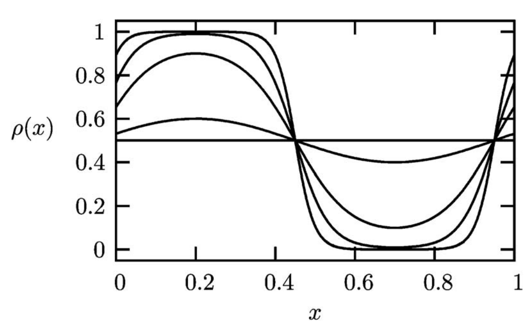

FIG. 2. The optimal profile for the WASEP when j_0=0 and ρ_0 =1/2. The different curves correspond to several values of ν (ν =6.34, 7.98, 12.0, 21.1). According to Eq. (17) the flat profile is optimal for $\nu < 2\pi \approx 6.28$.

$$v = \nu \frac{\sigma(g_2) - \sigma(g_1)}{g_2 - g_1}. \tag{31}$$

Then using the fact that g' vanishes when $g(x)=g_1$ or g_2 and replacing Eqs. (27) and (29)–(31) into Eq. (23) implies that the constants C_1, C_2 [in Eq. (23)] vanish at order ν^2. Thus asymptotically in ν, one can rewrite Eq. (23) as

$$g'^2 = \nu^2 \frac{1}{D(g)^2 (g_1 - g_2)^2} \{ g_2[\sigma(g_1) - \sigma(g)] + g_1[\sigma(g) - \sigma(g_2)] + g[\sigma(g_2) - \sigma(g_1)] \}^2 \tag{32}$$

and

$$G(j_0) = -\nu^2 \int \frac{dx}{\sigma(g)(g_1 - g_2)^2} \{ g_2[\sigma(g_1) - \sigma(g)] + g_1[\sigma(g) - \sigma(g_2)] + g[\sigma(g_2) - \sigma(g_1)] \}^2. \tag{33}$$

If one replaces g by its expression (28), one gets that the order ν^2 of $G(j_0)$ vanishes. The next order in the large ν expansion is dominated by the rounding-off of the discontinuities in (28) as given by (32). As the profile $g(x)$ is composed of two monotonic parts one can then use (32) into (33) and obtain for $g_2 > g_1$

$$G(j_0) = -2\nu \int_{g_1}^{g_2} dg \frac{D(g)}{(g_2 - g_1)\sigma(g)} |\sigma(g)(g_2 - g_1) - \sigma(g_1)(g_2 - g) - \sigma(g_2)(g - g_1)|.$$

It is remarkable that y is not present in this expression. In the case of the weakly asymmetric exclusion process on a ring, expressions (24) and (25) of $D(\rho)$ and $\sigma(\rho)$ lead for large ν to

$$G(j_0) = -\nu \left[g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} - (1-g_1)(1-g_2) \ln\left(\frac{1-g_1}{1-g_2} \right) \right]. \tag{34}$$

In this case (31) becomes $v = \nu(1 - g_1 - g_2)$.

If one takes formally $\nu=N$, the hopping rates $1/2 \pm \nu/2N$ become 1 and 0, so that the model reduces to the totally asymmetric exclusion process and one gets from Eqs. (7), (27), and (34)

$$\text{Pro}\left(\frac{Q_T}{T} = i_0 \right) \sim \exp\left\{ -T \left[g_2 - g_1 - g_1 g_2 \ln \frac{g_2}{g_1} - (1-g_1)(1-g_2) \ln\left(\frac{1-g_1}{1-g_2} \right) \right] \right\}. \tag{35}$$

As we will see in Sec. VII, this is exactly the large deviation function predicted by the Jensen-Varadhan theory [18] to maintain a profile (28) formed of a shock and an antishock in the totally asymmetric exclusion process. Other aspects of the relation between the large deviation functional of the weakly asymmetric exclusion process and the Jensen-Varadhan functional in systems with open boundary conditions will be presented in Ref. [23].

VII. LARGE DEVIATIONS OF THE CURRENT IN THE TOTALLY ASYMMETRIC EXCLUSION PROCESS

In the totally asymmetric simple exclusion process (TASEP), each particle jumps to its neighboring site, on its right, at rate 1, if the target site is empty (and there is no other jump).

The large deviations of the current of the totally asymmetric exclusion process on a ring of N sites, with P particles, have been calculated exactly [21,24]. If Q_T is the total number of jumps during time T over a given bond on the ring, one knows that for large T,

$$\langle e^{\lambda Q_T}\rangle \sim e^{\mu(\lambda)T} \tag{36}$$

and explicit expressions of $\mu(\lambda)$ have been obtained for all N and P by the Bethe ansatz [21,24].

For large N, it was shown in particular [Eq. (53) of Ref. [24] with the proper redefinition of the parameters] that for $\lambda<0$

$$\mu(\lambda) = -\frac{(1-e^{\lambda\rho_0})(1-e^{\lambda(1-\rho_0)})}{(1-e^{\lambda})}. \tag{37}$$

We are now going to argue that this result can be understood, by assuming that Eq. (36) is dominated by configurations of the form (28) moving at a velocity $v=1-g_1-g_2$. These density profiles are everywhere constant except for a shock [at some position z with $g(z-0)=g_1$ and $g(z+0)=g_2$ for $g_2>g_1$] and an antishock [at position $z+y$ with $g(z+y-0)=g_2$ and $g(z+y+0)=g_1$]. From the Jensen-Varadhan theory [18], the probability of maintaining such a shape moving at this velocity on the ring over a very long period of time T reduces to the probability of maintaining an antishock between the densities g_2 and g_1 moving at velocity v. The probability of the latter event, which we denote by $P_T(g_1,g_2)$ is given [18] by

$$P_T(g_1,g_2) \sim \exp\left\{-T\left[g_2-g_1-g_1g_2\ln\frac{g_2}{g_1} - (1-g_1)(1-g_2)\ln\left(\frac{1-g_1}{1-g_2}\right)\right]\right\}. \tag{38}$$

The corresponding integrated current Q_T is

$$Q_T = T[yg_1(1-g_1)+(1-y)g_2(1-g_2)]$$

since over a long period of time T a given bond spends a fraction y of the time at density g_1 and $1-y$ at density g_2.

Therefore, if the configurations of the form (28) dominate the large deviations of the current, one expects

$$\mu(\lambda) = \max_{y,g_1,g_2}\left\{\lambda[yg_1(1-g_1)+(1-y)g_2(1-g_2)] - \left[g_2-g_1 - g_1g_2\ln\frac{g_2}{g_1} - (1-g_1)(1-g_2)\ln\left(\frac{1-g_1}{1-g_2}\right)\right]\right\}, \tag{39}$$

where the maximum has to satisfy the constraint

$$\rho_0 = yg_1 + (1-y)g_2. \tag{40}$$

A calculation of the optimum in Eq. (39), with the constraint (40) leads to

$$g_1 = \frac{e^{\lambda}-e^{\lambda(1-\rho_0)}}{e^{\lambda}-1}; \quad g_2 = \frac{e^{\lambda\rho_0}-1}{e^{\lambda}-1}$$

and Eq. (39) becomes Eq. (37).

This shows that the result of the Bethe ansatz (37) can be physically understood in terms of an optimal profile which takes the form of the step function (28). The probability of maintaining this profile is given by the Jensen-Varadhan expression (38) which in fact is identical to Eq. (35) obtained, in the large ν limit, for the WASEP from the hydrodynamic large deviation theory.

That the fluctuations are due, in the strong asymmetric case, to configurations formed by a gas of shocks and antishocks has been already pointed out by Fogedby [28–30]. The calculation of this section shows that the large deviation of the current, in the range $\lambda<0$, can be understood quantitatively in terms of a single pair of shock-antishock. Whether the Fogedby theory would allow us to understand all the current fluctuations, including the range $\lambda>0$ where the expression of $\mu(\lambda)$ is more complicated [24] than Eq. (37), remains an interesting open question.

VIII. CONCLUSION

In the present work we have determined the limit of stability (16) and (17) of a flat profile for a diffusive lattice gas on a ring. This instability beyond which the optimal profile becomes modulated is of the same nature as the phase transition found for several other nonequilibrium systems [12,25–27]. As the calculation is based on a local stability analysis, one cannot of course exclude first order transitions, i.e., that the flat profile might become globally unstable. In a recent work [31], a sufficient condition on (D,σ) was obtained to ensure that there is no phase transition irrespective of the current j_0 and the density ρ_0. We did not find a simple way to compare this global stability condition to our local stability criterion (16).

In Sec. V, we have obtained numerical evidence that the macroscopic fluctuation theory predicts correctly the large deviation function of the current for the weakly asymmetric exclusion process. The numerical results are consistent with the second order phase transition predicted in Sec. III, and with a modulated density profile moving at a constant velocity as suggested in Sec. IV. These results could in principle be confirmed by solving the Bethe ansatz equations for the WASEP, since $\mu(\lambda)$ can be calculated exactly for the ring geometry [32,33].

It would be interesting to extend the results of the present work to the case of open boundary conditions. One difficulty is that the time-independent profile, found in Ref. [11], is much more complicated than the flat profile for the ring geometry, and we did not succeed so far to obtain the condition which would generalize Eqs. (16) and (17) for this open geometry.

Lastly, we noticed that the large deviation function (35) obtained for the weakly asymmetric diffusion process in the large drift limit is identical to the one predicted for a strong asymmetry by the Jensen-Varadhan theory (38) (see also Ref. [23]).

Despite this bridge between the large deviation function of the current of weakly and strongly asymmetric systems, and some recent results on zero-range processes [17], a theory of current fluctuations for strongly asymmetric lattice gas such as the ASEP with open boundary conditions remains an open problem.

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APPENDIX: A DERIVATION OF EQS. (4) and (5)

We present here a heuristic derivation of the hydrodynamic large deviations (4). Let us consider a system of N sites and decompose it into N/l boxes of l sites each. Let us define the density $\rho_i(t)$ in box i at time t and $q_i(t)$ the total number of particles transferred from box i to box $i+1$ during a time interval $t,t+\tau$ [this time τ should be large enough for the $q_i(t)$ to be a Gaussian characterized by its average and its variance as in Eqs. (1) and (3), but short enough compared to the characteristic time of variation of the densities $\rho_i(t)$].

In order to compute the probability of an evolution $\{q_i,\rho_i\}$ over the microscopic time interval $[0,T]$, we assume that the local currents $q_i(t)$ are Gaussian and independent. The mean and the variance are given by Eqs. (1) and (3) applied to a chain of length l,

$$\mathrm{Pro}(\{q_i,\rho_i\}) \sim \exp\left[-\sum_{k=1}^{T/\tau}\sum_{i=1}^{N/l}\frac{1}{\frac{2\sigma(\rho_i(k\tau))\tau}{l}}\left(q_i(k\tau)+D(\rho_i(k\tau))\frac{\rho_{i+1}(k\tau)-\rho_i(k\tau)}{l}\tau-\nu\frac{\sigma(\rho_i(k\tau))}{N}\tau\right)^2\right].$$

The factor ν/N comes from the weak asymmetry of the jumps. Clearly the conservation of the number of particles gives

$$\rho_i(s+\tau)=\rho_i(s)+\frac{q_{i-1}(s)-q_i(s)}{l}. \tag{A1}$$

Now if one takes a continuous limit by writing

$$\rho_i(s)=\rho\left(\frac{il}{N},\frac{s}{N^2}\right) \tag{A2}$$

and one defines a rescaled current by

$$q_i(s)=\frac{\tau}{N}j\left(\frac{il}{N},\frac{s}{N^2}\right) \tag{A3}$$

one gets for the probability of observing a trajectory $\{j,\rho\}$ over the microscopic time interval $[0,T]$

$$\mathrm{Pro}(\{j,\rho\}) \sim \exp\left[-N\int_0^{T/N^2}ds \times\int_0^1 dx\frac{\left[j(x,s)+D(\rho(x,s))\frac{d\rho(x,s)}{dx}-\nu\sigma(\rho(x,s))\right]^2}{2\sigma(\rho(x,s))}\right]$$

which is exactly Eqs. (4) and (5). Furthermore, Eq. (A1) with the scaling (A2) and (A3), leads to Eq. (6).

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